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Uniform Lipschitz Constants in Chebyshev Polynomial Approximation

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INTRODUCTION

Let M denote an n-dimensional Haar subspace of $C[a, b]$. It is well known [6] that in Chebyshev approximation, any $f \in C[a, b]$ has a strongly unique best approximation $B(f)$ from M, i.e. there exists a number $\gamma > 0$ such that

$$\|f - m\| \geqslant \|f - B(f)\| + \gamma \, \|B(f) - m\|$$

for all $m \in M$, where γ is taken to be the largest such number. Here $\gamma = \gamma(f, M, n)$.

There have been many results (see, e.g., [2, 5, 8]) on the existence of uniform strong unicity constants which are independent of n, f, or M; we are here concerned with uniformity with respect to f.

DEFINITION 1. A set $S \subseteq C[a, b]$ has a uniform strong unicity constant if there exists a number $\Gamma > 0$ such that for all $f \in S$ and all $m \in M$,

$$\|f - m\| \geqslant \|f - B(f)\| + \Gamma \, \|B(f) - m\|.$$

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E. W. Cheney [3] proved that whenever a function has a strongly unique best approximate, then the best approximation operator B satisfies a Lipschitz condition at f, i.e., there exists a number $\lambda(f) > 0$ such that

$$\|B(f) - B(g)\| \leqslant \lambda(f)\, \|f - g\|$$

for all $g \in C[a, b]$, where $\lambda(f)$ is taken to be the smallest such number. In fact, the proof shows that $\lambda \leqslant 2/\gamma$.

DEFINITION 2. A set $S \subseteq C[a, b]$ has a uniform Lipschitz constant if there exists a number Λ such that for all $f \in S$ and all $g \in C[a, b]$,

$$\|B(f) - B(g)\| \leqslant \Lambda\, \|f - g\|.$$

From the proof of E. W. Cheney's result in [3] it follows that:

THEOREM 1 (Cheney). *If $S \subseteq C[a, b]$ has a uniform strong unicity constant Γ, then S has a uniform Lipschitz constant Λ and one can take $\Lambda \leqslant 2/\Gamma$.*

For a set $S \subseteq C[a, b]$, the results in [2] characterize the existence of a uniform strong unicity constant in terms of limit extreme point sets. Whenever there is a uniform strong unicity constant, there is by Theorem 1 a uniform Lipschitz constant. This paper considers the situation when there is not a uniform strong unicity constant. In particular, Theorem 7 completely determines for a bounded set of functions which have no "almost alternation sets" (see Definition 5) whether or not there is a uniform Lipschitz constant.

It is known [1] that when M is not necessarily a Haar set, if $B(\cdot)$ satisfies a Lipschitz condition at f and $B(f)$ is unique, then $B(f)$ is strongly unique. In this paper's circumstances, with M a Haar set, $B(f)$ will be unique, and it is surprising that, as seen in Example 1, if S has a uniform Lipschitz constant then it need not have a uniform strong unicity constant. This shows moreover, that there can be no inequality like $c/\gamma \leqslant \lambda$ for some $c > 0$ corresponding to the inequality $\lambda \leqslant 2/\gamma$ of E. W. Cheney. Theorem 5(b)(ii), combined with the results in [2], gives general conditions when there is no uniform strong unicity constant, but there is a uniform Lipschitz constant. In case dim $M = 1$, it is known [5] that there is a uniform strong unicity constant and hence a uniform Lipschitz constant. Hence throughout the paper it is assumed that $n \geqslant 2$.

A major difficulty in the study of (uniform) Lipschitz constants is the lack of characterizations for Lipschitz constants similar to the characterizations available for strong unicity constants.

2. Definitions and Notation

Henceforth $M = \prod_{n-1}$, the algebraic polynomials of degree less than or equal to $n-1$. The best approximate from M to a given $f \in C[a, b]$ is denoted by Bf or $B(f)$. The extreme point set for f is

$$E(f) = \{x : |(f - Bf)(x)| = \|f - Bf\|\}$$

and the positive (resp. negative) points $E^+(f)$ (resp. $E^-(f)$) are those points $x \in E(f)$ where $(f - Bf)(x) = \|f - Bf\|$ (resp. $(f - Bf)(x) = -\|f - Bf\|$). Say the sign of a point in $E^+(f)$ or $E^-(f)$ is $+$ or $-$, respectively. Let $A(f)$ denote an alternation set of $n+1$ points for f. For a finite set D denote its separation by sep $D = \min\{|x - y| : x, y \in D\}$. The concept of a limit extreme point was important in studying uniform strong unicity constants in [2].

Definition 3. Let $S = \{f_k\}$ be a sequence of functions in $C[a, b]$. A point $x \in [a, b]$ is called a $+$ *limit extremal* of S if for each k there exists $x_k^+ \in E^+(f_k)$ such that $\lim_{k \to \infty} x_k^+ = x$. A $-$ *limit extremal* is defined similarly. A point $x \in [a, b]$ is a $\pm$ *limit extremal* of S if for each k there exists $x_k^+ \in E^+(f_k)$ and $x_k^- \in E^-(f_k)$ such that $\lim_{k \to \infty} x_k^+ = \lim_{k \to \infty} x_k^- = x$. Denote the three sets of these limit extremals by $\mathrm{LE}^+(S)$, $\mathrm{LE}^-(S)$, and $\mathrm{LE}^{\pm}(S)$, respectively.

In general reference to the convergence of subsets of $[a, b]$ refers to convergence of sets in the compact metric space consisting of the nonempty, closed subsets of $[a, b]$ with the Hausdorff metric. For subsets $A, B \subseteq [a, b]$ the Hausdorff metric $d(A, B)$ is defined by

$$d(A, B) = \max\{\max_{a \in A} \min_{b \in B} |a - b|, \max_{b \in B} \min_{a \in A} |a - b|\}.$$

Note that if $\{A_k\} \to A$, then $A = \{\lim x_k : x_k \in A_k \text{ for all } k\}$.

Definition 4. Let $\{f_k\} \subseteq C[a, b]$ satisfy $\{E(f_k)\} \to E^0$. Then E^0 is called a *limit extreme point set*. If $\{A(f_k)\} \to A^0$ for some choice of $A(f_k)$, call A^0 a *limit alternation set* if $|A^0| = n+1$.

In addition to the above ideas in [2], we also use the idea of an almost alternation set. Example 2 in [7] can be interpreted as an example of a family of functions S which has an almost alternation set. In that example, extended to $[-1, 1]$ and normalized so that $Bf_k = 0$ and $\|f_k\| = 1$, $\{f_k\}$ is a sequence of functions such that $f_k(w_k) > -1$ for some $w_k \in [a, b]$ and $\lim_{k \to \infty} f_k(w_k) = -1$ so that w_k is almost an alternation point for f_k. First let the "ordered distance" between alternation sets $A(f) = \{x_i\}_{i=1}^{n+1}$ and

$A(g)=\{y_i\}_{i=1}^{n+1}$ be denoted by $d_0(A(f), A(g))=\max_{1\leq i\leq n}|x_i-y_i|$ when $x_1\in E^+(f)$ and $y_1\in E^+(g)$, or when $x_1\in E^-(f)$ and $y_1\in E^-(g)$; otherwise $d_0(A(f), A(g))$ is set equal to $b-a$.

Remark. In fact, if we say that two alternation sets $A(f)$ and $A(g)$ are equal if both $x_i=y_i$, $i=1, ..., n+1$ and they alternate the same way, i.e., $x_1\in E^+(f)$ implies $y_1\in E^+(g)$ and $x_1\in E^-(f)$ implies $y_1\in E^-(g)$, then the ordered distance is a distance function on the set of alternation sets. Furthermore, this ordered distance is always at least as large as the Hausdorff distance.

DEFINITION 5. A sequence $S=\{f_k\}_{k=1}^{\infty}$ does not have an *almost alternation set* if whenever a sequence $\{g_k\}_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty}\|g_k-f_k\|=0$ there is a constant M such that for all $k=1, ...,$

$$d_0(A(f_k), A(g_k))\leq M\,\|f_k-g_k\|,$$

where $A(f_k)$ and $A(g_k)$ are any alternation sets for f_k and g_k, respectively.

It follows from the definition that when there is no almost alternation set and $\{g_k\}$ is a sequence such that $\lim_{k\to\infty}\|f_k-g_k\|=0$, then $\lim_{k\to\infty} d_0(A(f_k), A(g_k))=0$, and if in addition $\{A(f_k)\}\to A^0$ then $\lim_{k\to\infty} d(A(g_k), A^0)=0$ because $\lim_{k\to\infty} d(A(f_k), A^0)=0$.

Remark. If there is a uniform Lipschitz constant for $S\subseteq C[a, b]$, it is easy to show using (3.20) below that if w is a minus limit extremal for $\{f_k\}\subseteq S$ with $\|f_k\|=1$, $Bf_k=0$, and $f_k(x)\leq 1-\eta$ in some neighborhood N of w with $0<\eta<1$, and thus there are no + extremals for f_k in N, then any sequence $\{g_k\}$ in $C[a, b]$ with $\|g_k-f_k\|=\delta_k\downarrow 0$ can not have plus extremals $\{w_k\}$ in N for large k. This follows because from (3.20), $\|Bg_k-g_k\|\to 1$ and since there is by assumption a uniform Lipschitz constant $\|Bf_k-Bg_k\|\leq A\,\|f_k-g_k\|$ which implies $\{\|Bg_k\|\}\to 0$ and thus $\|g_k\|\to 1$. If w_k is a plus extremal for g_k then $\{g_k(w_k)\}\to 1$ which contradicts $f\mid N\leq 1-\eta$. Hence the existence of a uniform Lipschitz constant in some cases requires that

$$\lim_{k\to\infty} d_0(A(f_k), A(g_k))=0.$$

If, on the other hand, $\sup_{x\in N} f_k(x)=1-\eta_k$, $\eta_k>0$, and $\lim_{k\to\infty}\eta_k=0$, then those points $x_k\in N$, where $f_k(x_k)=1-\eta_k$ can be used to construct a sequence $\{g_k\}$ such that $\|f_k-g_k\|\to 0$ and the other condition of Definition 5 is violated.

Since the pattern for uniform Lipschitz constants resembles and makes use of that for uniform strong unicity constants we state those results here.

THEOREM 2[2]. *A set $S \subseteq C[a,b]\backslash M$ does not have a uniform strong unicity constant if and only if S contains a sequence $\{f_k\}$ with $\{E(f_k)\} \to E^0$ where one of the following holds:*

(i) $|E^0| \leqslant n-1$

(ii) $|E^0| = n$ *and* E^0 *contains a point which is not a* $\pm$ *limit extremal of* $\{f_k\}$.

(iii) $|E^0| \geqslant n+1$ *and* E^0 *does not contain a limit alternation set for any subsequence of* $\{f_k\}$.

In [2, Theorems 5 and 6], it was shown that in the two cases when a sequence $\{f_k\}$ with $\{E(f_k)\} \to E^0$, satisfies E^0 contains at least n $\pm$ limit extremals, or when $|E^0| \geqslant n+1$ and E^0 contains a limit alternation set, then $\{f_k\}$ has a uniform strong unicity constant. Hence by Theorem 1 there would also be a uniform Lipschitz constant.

We should observe finally that in order for $S \subseteq C[a,b]$ to have a uniform Lipschitz constant it is necessary and sufficient that every sequence from S has a uniform Lipschitz constant.

3. RESULTS

First we state and prove Theorems 3, 5, and 6, the main theorems of the paper. Next in Section 4 we give Example 1 that illustrates the strange behavior of Lipschitz constants when there is an almost alternation set, shows how sensitive Lipschitz constants are to small changes in extreme point sets, and illustrates what happens in Theorems 5 and 6 in case there is an almost alternation set.

THEOREM 3. *If $S = \{f_k\}$ is a sequence in $C[a,b]\backslash M$, $\{E(f_k))\} \to E^0$ and $|E^0| \leqslant n-1$, $n \geqslant 2$, then S does not have a uniform Lipschitz constant.*

Proof. Let $E^0 = \{a_1, ..., a_L\}$, $1 \leqslant L \leqslant n-1$, be the limit of $\{E(f_k)\}_{k=1}^{\infty}$. Let $\operatorname{sep}\{a_i\} = \delta$ so that $|a_i - a_j| \geqslant \delta$ for $i \neq j$. We can assume without loss of generality that for each k, $\|f_k\| = 1$ and $B(f_k) = 0$. Define

$$O_m = \{x \in [a,b] : |x - a_l| < 1/m \text{ for some } l = 1, ..., L\}$$

for $1/m < \delta/4$. Then O_m is a union of L disjoint open (in $[a,b]$) intervals with $E^0 \subset O_m$. Also for k large enough, $k > k(m)$, $E(f_k) \subseteq O_m$. Since

$E(f_k) \subseteq O_m$, $\|f_k\|_{O_m^c} < 1$. Hence let $\|f_k\|_{O_m^c} = 1 - \delta_k$, $\delta_k > 0$. Then there exists a polynomial $p_k \in M$ such that

$$\begin{aligned} &\text{(i)} && p_k(a_i) = 0, \qquad i = 1, ..., L \\ &\text{(ii)} && \|p_k\| = \delta_k \\ &\text{(iii)} && p_k \text{ has no other zeros.} \end{aligned} \tag{3.1}$$

Let $V_m = \{x \in [a, b]: |x - a_l| < 1/m + \delta/4$, for some $l = 1, ..., L\}$. Thus V_m consists of disjoint, open (in $[a, b]$) intervals and $O_m \subseteq V_m$. Define $w_k(x)$ on $\bar{O}_m \cup V_m^c$ by

$$w_k(x) = \begin{cases} 0, & \text{if } x \in V_m^c \\ p_k(x), & \text{if } x \in \bar{O}_m. \end{cases} \tag{3.2}$$

Then on $\bar{O}_m \cup V_m^c$, w_k satisfies

$$|w_k(x) - p_k(x)| \leqslant |p_k(x)|. \tag{3.3}$$

By the Tietze Extension Theorem, w_k can be extended to all of $[a, b]$ such that (calling the extended function $w_k(x)$) $w_k(x)$ satisfies (3.3) on all of $[a, b]$ and for any $x \in [a, b]$

$$|w_k(x)| \leqslant \|w_k\|_{\bar{O}_m \cup V_m^c} = \|p_k\|_{\bar{O}_m}. \tag{3.4}$$

Now define $g_k(x) \in C[a, b]$ to be $g_k(x) = f_k(x) + w_k(x)$. First we have $\|g_k - f_k\| = \|w_k\| = \|p_k\|_{\bar{O}_m}$. Then it is shown that $B(g_k) = p_k$ follows from (3.1)–(2.4). We have

$$\begin{aligned} \|g_k - p_k\|_{V_m^c} &= \|f_k - p_k\|_{V_m^c} \\ &\leqslant \|f_k\|_{O_m^c} + \|p_k\| \\ &= 1. \end{aligned}$$

Also

$$\|g_k - p_k\|_{O_m} = \|f_k\|_{O_m} = 1,$$

and if $x \in O_m^c \setminus V_m^c$,

$$|(g_k - p_k)(x)| \leqslant |f_k(x)| + |(w_k - p_k)(x)| \leqslant 1.$$

Finally if $x_j \in E(f_k) \subseteq O_m$, then since $w_k = p_k$ on $\bar{O}_m$, $(g_k - p_k)(x_j) = f_k(x_j) = \pm 1$ alternately at at least $n + 1$ points. Thus $B(g_k) = p_k$ and $\|g_k - p_k\| = 1$. Now

$$\frac{\|B(f_k) - B(g_k)\|}{\|f_k - g_k\|} = \frac{\|p_k\|}{\|p_k\|_{\bar{O}_m}} \tag{3.5}$$

By the Mean Value Theorem, if $x \in \bar{O}_m$,

$$p_k(x) = p_k(x) - p_k(a_l) = p'_k(\eta)(x - a_l) \tag{3.6}$$

for some η and l, and Markoff's inequality implies

$$\| p'_k \| \leqslant (n-1)^2 \, \| p_k \|. \tag{3.7}$$

Thus from (3.6) and (3.7) for some $l = 1, ..., L$

$$\begin{aligned} |p_k(x)| &\leqslant (n-1)^2 \, \|p_k\| \, |x - a_l| \\ &\leqslant (n-1)^2 \, \|p_k\|/m. \end{aligned} \tag{3.8}$$

Hence

$$\frac{\|p_k\|}{\|p_k\|_{\bar{O}_m}} \geqslant \frac{\|p_k\|}{(n-1)^2 \, \|p_k\|/m} = \frac{m}{(n-1)^2}. \tag{3.9}$$

By (3.5) and (3.9) then $\lambda(f_k) \geqslant m/(n-1)^2$. Since this holds for any sufficiently large m, it follows that

$$\sup\{\lambda(f_k): k = 1, ...\} = \infty$$

and the proof is complete.

The following Lemma shows that the existence of a uniform Lipschitz constant for a bounded set $S \subseteq C[a, b]$ depends on the behavior of functions $g \in C[a, b]$ which are close to functions $f \in S$.

LEMMA 4. *If* $\lim_{k \to \infty} \lambda(f_k) = \infty$, *and if* $\{f_k\}$ *is bounded, then there exists a sequence of functions* $\{g_k\}$ *from* $C[a, b]$ *such that*

(i) $$\lim_{k \to \infty} \|g_k - f_k\| = 0$$

(ii) $$\lim_{k \to \infty} \frac{\|B(f_k) - B(g_k)\|}{\|f_k - g_k\|} = \infty.$$

Proof. Since $\lim_{k \to \infty} \lambda(f_k) = \infty$, there exists by the definition of λ a sequence of functions $\{g_k\}$ such that

$$\lim_{k \to \infty} \frac{\|Bf_k - Bg_k\|}{\|f_k - g_k\|} = \infty.$$

Then (i) follows because if there were to exist an $\varepsilon > 0$ such that $\|f_k - g_k\| > \varepsilon$ for all k, then

$$\begin{aligned}\|B(f_k) - B(g_k)\| &= \|B(g_k - B(f_k))\| \\ &\leqslant 2\,\|g_k - B(f_k)\| \\ &\leqslant 2(\|g_k - f_k\| + \|f_k - B(f_k)\|)\end{aligned}$$

and hence

$$\begin{aligned}\frac{\|B(f_k) - B(g_k)\|}{\|f_k - g_k\|} &\leqslant 2 + 2\,\frac{\|f_k - B(f_k)\|}{\|g_k - f_k\|} \\ &\leqslant 2 + 2\,\|f_k - B(f_k)\|/\varepsilon,\end{aligned}$$

which is bounded.

Theorems 3 and 6 consider situations when $|E^0| \leqslant n-1$ or $|E^0| \geqslant n+1$, respectively. Theorem 5 is concerned, essentially, with the case $|E^0| = n$.

THEOREM 5. *Let* $S = \{f_k\}$ *be a sequence in* $C[a, b]\setminus M, n \geqslant 2$ *and* $\{E(f_k)\} \to E^0$.

(a) *If* $|\mathrm{LE}^{+}\ (S)| \geqslant n$
then S has a uniform Lipschitz constant.

(b) *Suppose* $|E^0| = n$ *and* $\{A(f_k)\} \to A^0$.
   (i) *If* $E^0\setminus(A^0 \cup \mathrm{LE}^{+-}\ ) \neq \varnothing$ *then there is no uniform Lipschitz constant.*
   (ii) *If* $|A^0| = n$, *there exists no almost alternation set, and S is bounded, then there is a uniform Lipschitz constant.* (3.10)

Proof. Part (a) follows immediately from [2, Theorem 5] and Theorem 1. First we prove (b)(i) in a manner similar to the proof of Theorem 3.

Let $E^0 = \{a_1, ..., a_n\}$, where $a_n \in E^0\setminus(A^0 \cup \mathrm{LE}^{+}\ \ )$ so $A^0 \subseteq \{a_1, ..., a_{n-1}\}$. Also let $\delta = \mathrm{sep}\{a_i\}$. Let

$$O_m = \{x \in [a, b]: |x - a_l| < 1/m \text{ for some } l = 1, ..., n-1\}, \tag{3.11}$$

where $1/m < \delta/4$, and

$$U_m = \{x \in [a, b]: |x - a_n| < 1/m\}. \tag{3.12}$$

For k sufficiently large, $E(f_k) \subseteq O_m \cup U_m$ and since $a_n \notin \mathrm{LE}^{+}\ $, we can assume (without loss of generality) $a_n \notin \mathrm{LE}\ \ $ so there exists a neighborhood V of a_n and $c_k > 0$ for k sufficiently large such that if $x \in V$, then

$$f_k(x) \geqslant -1 + c_k. \tag{3.13}$$

Assume m is so large that $U_m \subseteq V$. There exists a $\delta_k > 0$ such that

$$\|f_k\|_{[O_m \cup U_m]^c} = 1 - \delta_k. \tag{3.14}$$

Let

$$V_m = \{x \in [a, b]: |x - a_l| < 1/m + \delta/4 \text{ for some } l = 1, ..., n-1\}.$$

Thus $O_m \subseteq V_m$ and $U_m \subseteq V_m^c$. Define $p_k(x) \in M$ by

(i) $p_k(a_i) = 0, \; i = 1, ..., n-1$

(ii) $p_k(a_n) > 0$

(iii) $\|p_k\| = \min\{c_k, \delta_k\}$,

where we can assume $p_k \mid U_m > 0$. Define $w_k \in C[a, b]$ by

$$w_k(x) = \begin{cases} p_k(x), & x \in \bar{O}_m \\ 0, & x \in V_m^c \end{cases}$$

so that on $\bar{O}_m \cup V_m^c$, w_k satisfies

$$|w_k(x) - p_k(x)| \leqslant |p_k(x)| \tag{3.15}$$

and then extend $w_k(x)$ to all of $[a, b]$ so that (3.15) holds on all of $[a, b]$ and if $x \in [a, b]$,

$$|w_k(x)| \leqslant \|w_k\|_{\bar{O}_m \cup V_m^c} = \|p_k\|_{\bar{O}_m}. \tag{3.16}$$

Now let $g_k = f_k + w_k$. As in Theorem 3, considering $\|g_k - p_k\|$ in turn on O_m, U_m, $O_m^c \backslash V_m^c$, $(U_m \cup V_m)^c$ and at points $x_j \in A(f_k) \subseteq O_m$ we obtain $B(g_k) = p_k$. This leads as before to

$$\sup \lambda(f_k) = \infty,$$

and the proof of (b)(i) is completed.

To prove (b)(ii) assume to the contrary that some sequence $\{f_k\}$ of functions from S satisfies $\{\lambda(f_k)\} \uparrow \infty$. Since $\{f_k\}$ has no almost alternation set and S is bounded, $\{f_k/\|f_k\|\}$ will have no almost alternation set. Thus, without loss of generality we assume $\|f_k\| = 1$ and $Bf_k = 0$ for each k. Let sep $A^0 = \eta$ and let ε satisfy $0 < \varepsilon < \eta/8$. By Lemma 4 let $\{g_k\}$ be a sequence of functions from $C[a, b]$ with

$$\|g_k - f_k\| = \delta_k \downarrow 0 \tag{3.17}$$

and

$$\|Bf_k - Bg_k\|/\delta_k \uparrow \infty.$$

Since $\lim_{k\to\infty} d(A(f_k), A^0) = 0$, there is a constant K such that $k > K$ implies

$$d(A(f_k), A^0) < \varepsilon. \tag{3.18}$$

By the definition of an almost alternation set there is some constant M such that for all k,

$$d_0(A(g_k), A(f_k)) \leqslant M \| f_k - g_k \|. \tag{3.19}$$

We know that

$$1 - \delta_k \leqslant \| Bg_k - g_k \| \leqslant 1 + \delta_k \tag{3.20}$$

since

$$\| Bg_k - g_k \| \leqslant \| g_k \| \leqslant \| g_k - f_k \| + \| f_k \| \leqslant 1 + \delta_k$$

and

$$\begin{aligned} 1 - \| Bg_k - g_k \| &= \| f_k \| - \| Bg_k - g_k \| \\ &\leqslant \| f_k - Bg_k \| - \| Bg_k - g_k \| \\ &\leqslant \| f_k - g_k \| \\ &\leqslant \delta_k. \end{aligned}$$

Let w_k denote one of the n points in A^0. By (3.18) there is a point $x(f_k) \in A(f_k)$ with

$$| x(f_k) - w_k | < \varepsilon. \tag{3.21}$$

Assume that $x(f_k) \in E^-(f_k)$, $\lim_{k\to\infty} x(f_k) = w_k$, and $w_k \in \mathrm{LE}^-(\{f_k\})$. (The case $x(f_k) \in E^+(f_k)$ is similar.) By (3.19) there is a point $x(g_k) \in A(g_k)$ such that $x(g_k) \in E^-(g_k)$ and

$$| x(g_k) - x(f_k) | \leqslant M\delta_k, \tag{3.22}$$

where we assume without loss of generality that $M\delta_k < \eta/8$.

From (3.20) follows

$$-g_k(x(f_k)) + Bg_k(x(f_k)) \leqslant 1 + \delta_k \tag{3.23}$$

and hence from (3.17) follows

$$\begin{aligned} Bg_k(x(f_k)) &\leqslant 1 + \delta_k + g_k(x(f_k)) \\ &\leqslant 2\delta_k. \end{aligned} \tag{3.24}$$

Also from (3.20) follows

$$g_k(x(g_k)) - Bg_k(x(g_k)) = -\|g_k - Bg_k\| \leqslant -1 + \delta_k$$

and hence from (3.17) follows

$$\begin{aligned} Bg_k(x(g_k)) &\geqslant -\delta_k + 1 + g_k(x(g_k)) \\ &\geqslant -2\delta_k. \end{aligned} \tag{3.25}$$

Moreover since $\|Bg_k\| \leqslant 2\, \|g_k\| \leqslant 3$ (assuming $\delta_k < \frac{1}{2}$), by Markoff's inequality and (3.22) for some $z \in [a, b]$,

$$\begin{aligned} -Bg_k(x(f_k)) + Bg_k(x(g_k)) &\leqslant |Bg_k(x(f_k)) - Bg_k(x(g_k))| \\ &\leqslant |(Bg_k)'(z)|\, |x(f_k) - x(g_k)| \\ &\leqslant 3(n-1)^2\, M\delta_k, \end{aligned}$$

and hence by (3.24)

$$Bg_k(x(g_k)) \leqslant \delta_k(2 + 3M(n-1)^2). \tag{3.26}$$

Hence by (3.25) and (3.26) there is a constant M_1, independent of k, such that

$$|Bg_k(x(g_k))| \leqslant M_1 \delta_k. \tag{3.27}$$

Since $M\delta_k + \varepsilon < \eta/4$, there is one such point $x(g_k)$ satisfying (3.27) corresponding to each one of the points in A^0; denote these points by w_i, $i = 1, ..., n$.

Then by the Lagrange form of the interpolating polynomial Bg_k, for each $x \in [a, b]$,

$$\begin{aligned} |Bg_k(x)| &= \left| \sum_{i=1}^{n} Bg_k(w_i) \prod_{j=1, j \neq i}^{n} (x - w_j) \Big/ \prod_{j=1, j \neq i}^{n} (w_i - w_j) \right| \\ &\leqslant \sum_{i=1}^{n} M_1 \delta_k (b-a)^{n-1} / (\eta/2)^{n-1} \\ &\leqslant M_2 \delta_k \end{aligned}$$

for some constant M_2 independent of k. Thus

$$\frac{\|Bf_k - Bg_k\|}{\|f_k - g_k\|} = \frac{\|Bg_k\|}{\delta_k} \leqslant M_2.$$

This contradiction completes the proof.

THEOREM 6. *Let* $S = \{f_k\}$ *be a sequence in* $C[a, b] \setminus M$ *with* $\{E(f_k)\} \to E^0$.

(a) *If* E^0 *contains a limit alternation set, then* S *has a uniform Lipschitz constant.*

(b) *If* E^0 *with* $|E^0| \geqslant n+1$ *is not a limit alternation set, then there always exists an almost alternation set.*

Proof. Part (a) follows from [2, Theorem 6] and Theorem 1. To prove part (b) as usual we can assume that $\|f_k\| = 1$ and $Bf_k = 0$ for all k. We claim that for some subsequence $\{f_{k_i}\}$ of S and some choice of $\{A(f_{k_i})\}$ we have $\{A(f_{k_i})\} \to B^0$ with $|B^0| < |E^0|$. If not, choosing $\{A(f_k)\}$ arbitrarily and choosing any convergent subsequence $\{A(f_{j_i})\}$ we would have $\{A(f_{j_i})\} \to$ some C^0 with $|C^0| \geqslant |E^0|$, so $|C^0| = |E^0| = n+1$ and $C^0 = E^0$; but then $\{A(f_k)\} \to E^0$ (else for some other subsequence $\{A(f_{l_i})\}$ of $\{A(f_k)\}$ we would have $d(A(f_{l_i}), E^0) \geqslant$ some $\varepsilon > 0$, and some subsequence of $\{A(f_{l_i})\}$ would converge to some B^0 with necessarily $|B^0| < |E^0|$), so E^0 would itself be a limit alternation set, contrary to assumption. We now show that $\{A(f_{k_i})\} \to B^0$ with $|B^0| < |E^0|$ implies that there is an almost alternation set for S. Let $B^0 = \{e_1, ..., e_s\}$ and suppose $e_{s+1} \in E^0 \setminus B^0$, where we assume that for some l we have $e_l < e_{s+1} < e_{l+1}$. (The cases $e_{s+1} < \min\{e_1, ..., e_s\}$ and $e_{s+1} > \max\{e_1, ..., e_s\}$ are similar.) Going to further subsequences if necessary, for all i we find $e_l^{(k_i)}, e_{l+1}^{(k_i)}$ adjacent points in $A(f_{k_i})$, $e_{s+1}^{(k_i)} \in E(f_{k_i})$ with $e_l^{(k_i)} \to e_l$, $e_{l+1}^{(k_i)} \to e_{l+1}$, $e_{s+1}^{(k_i)} \to e_{s+1}$; without loss of generality we can also assume $f_{k_i}(e_l^{(k_i)}) = f_{k_i}(e_{s+1}^{(k_i)}) = 1$ and $f_{k_i}(e_{l+1}^{(k_i)}) = -1$ for all i. Now let $g_k = f_k$ for all k, with $A(g_k)$ taken to be $A(f_k)$, except that for all i sufficiently large to ensure $e_l^{(k_i)} < e_{s+1}^{(k_i)} < e_{l+1}^{(k_i)}$ replace $e_l^{(k_i)}$ by $e_{s+1}^{(k_i)}$ to form $A(g_{k_i})$. For all such i we will have $d_0(A(g_{k_i}), A(f_{k_i})) = e_{s+1}^{(k_i)} - e_l^{(k_i)} > 0$, violating Definition 5.

The following theorem summarizes the results in the case of a bounded set of functions with no almost alternation sets.

THEOREM 7. *Let* $S \subseteq C[a, b] \setminus M$ *be bounded and assume no sequence in* S *has an almost alternation set. Then* S *does not have a uniform Lipschitz constant if and only if there is a sequence* $\{f_k\} \subseteq S$ *such that* $\{E(f_k)\} \to E^0$ *and* $|E^0| \leqslant n-1$.

Proof. If there is a sequence $\{f_k\} \subseteq S$ with $\{E(f_k)\} \to E^0$ and $|E^0| \leqslant n-1$, then by Theorem 3, $\{f_k\}$ (and thus S) does not have a uniform Lipschitz constant. If on the other hand S does not have a uniform Lipschitz constant, then there is a sequence $\{f_k\} \subseteq S$ with $\lambda(f_k) \to \infty$; by going to a subsequence if necessary we can assume $\{E(f_k)\} \to$ some E^0 and $\{A(f_k)\} \to$ some A^0. Then $|E^0| \geqslant n+1$ is impossible by Theorem 6, $|E^0| = n$ and $|A^0| = n$ is impossible by Theorem 5(b)(ii), and $|E^0| = n$ and

$|A^0| < n$ is impossible since by the proof of Theorem 6(b) that would imply that there is an almost alternation set. Thus $|E^0| \leqslant n-1$.

The degeneracy of A^0 in Theorem 5(b)(i) where $|A^0| \leqslant n-1 < |E^0| = n$ guaranteed the nonexistence of a uniform Lipschitz constant, at least if $E^0 \setminus (A^0 \cup \mathrm{LE}^{+-}) \neq \varnothing$. However, when $|E^0| \geqslant n+1$ then the degeneracy of some A^0, i.e., $|A^0| \leqslant n-1$, may occur even when there is a uniform Lipschitz constant. See Example 1 below.

4. An Example

The following Example shows that in Theorem 5(b)(ii), if there is an almost alternation set then sometimes there is and sometimes there is not a uniform Lipschitz constant. Also in Theorem 6 part (b) the Example shows that in this case again there may or may not be a uniform Lipschitz constant because of the existence of the almost alternation set. Furthermore, by part (ii) of Theorem 2, the sequence of functions does not have a uniform strong unicity constant even when there is a uniform Lipschitz constant.

Example 1. Let $M = \Pi_1$, the algebraic polynomials of degree one or less on $[-1, 1]$ and let $f_{\alpha\beta\gamma c}(x) \in C[-1, 1]$ be defined on knots by

$$f_{\alpha\beta\gamma c}(x) = \begin{cases} -1 & \text{if } x = -1, 1 \\ 1 & \text{if } x = -1+\alpha \\ -1+\gamma & \text{if } x = -1+c \\ 1-\beta & \text{if } x = 0 \end{cases}$$

and be piecewise linear between these knots. It is assumed that $0 < \alpha < c \leqslant \frac{1}{2}$, $0 \leqslant \beta < 1$, and $0 < \gamma < 1$. Letting α and γ tend to 0 through a sequence of values let $S = \{f_{\alpha,\gamma}\}$. Then S has an almost alternation set. If β/γ is bounded, S has a uniform Lipschitz constant, whereas otherwise there might be no uniform Lipschitz constant. In this example, $E^0 = \{-1, 1\}$, 1 is not a $\pm$ limit extremal and $|A^0| = |E^0| = n$. Denote $f_{\alpha\beta\gamma c}$ by f.

To see that there is an almost alternation set, let $g(x)$ be defined to be equal to f on the knots except $g(-1+c) = -1$. Then $B(g) = 0$, $A(f) = \{-1, -1+\alpha, 1\}$, we can take $A(g) = \{-1, -1+\alpha, -1+c\}$, $\|f-g\| = \gamma$ and

$$\frac{d_0(A(f), A(g))}{\|f-g\|} = \frac{2-c}{\gamma}$$

which does not remain bounded as $\gamma \to 0$.

Now for any values of α, β, γ, c we derive some inequalities arising from the values of another function g at the knots. Let $\|f-g\|=\delta$. Then (3.20) shows that

$$1-\delta \leqslant \|B(g)-g\| \leqslant 1+\delta. \tag{4.1}$$

Using (4.1) we obtain

$$\begin{aligned} Bg(-1) &\leqslant 2\delta \\ Bg(-1+\alpha) &\geqslant -2\delta \\ Bg(-1+c) &\leqslant 2\delta+\gamma \\ Bg(0) &\geqslant -2\delta-\beta \\ Bg(1) &\leqslant 2\delta. \end{aligned} \tag{4.2}$$

Now suppose (by way of contradiction) that β/γ is bounded, but the sequence of functions f does not have a uniform Lipschitz constant; going to a subsequence if necessary we can assume $\lambda(f)\to\infty$. Then by Lemma 4 we can assume $\delta\to 0$ and $\|Bg\|/\delta\to\infty$. Let m be the slope of Bg; we next compute bounds for $|m|$ and $\|Bg\|$. Since the graph of Bg is a straight line, it follows from $Bg(-1)\leqslant 2\delta$ and $Bg(1)\leqslant 2\delta$ that $Bg(x)\leqslant 2\delta$, for all $x\in[-1,1]$. From (4.2) we also have $m=(Bg(0)-Bg(-1))/(0-(-1))\geqslant -\beta-2\delta-2\delta=-\beta-4\delta$, so $Bg(1)=Bg(0)+(1-0)m\geqslant -\beta-2\delta-\beta-4\delta=-2\beta-6\delta$. Further, $m=(Bg(1)-Bg(-1+\alpha))/(1-(-1+\alpha))\leqslant(2\delta-(-2\delta))/(2-\alpha)=4\delta/(2-\alpha)$, so $Bg(-1)=Bg(-1+\alpha)+(-1-(-1+\alpha))m \geqslant -2\delta-4\delta\alpha/(2-\alpha)$. Thus since $\alpha<\frac{1}{2}$ we have $|m|\leqslant\max(\beta+4\delta, 4\delta/(2-\alpha))=\beta+4\delta$, and $\|Bg\|\leqslant\max(2\delta, 2\beta+6\delta, 2\delta+4\delta\alpha/(2-\alpha))=2\beta+6\delta$, i.e.,

$$\begin{aligned} |m| &\leqslant \beta+4\delta \\ \|Bg\| &\leqslant 2\beta+6\delta. \end{aligned} \tag{4.3}$$

Now $\beta\to 0$, $\delta\to 0$, and $\gamma\to 0$, so we can assume $|m|$ and $\|Bg\|$ are as small as we like, so we can assume $|m|$ is smaller than the absolute values of the slopes of the four line segments comprising f, and if the zeroes of f are denoted by z_1, z_2, z_3, z_4 then we can assume the minus extremals of g lie in $[-1, z_1)\cup(z_2, z_3)\cup(z_4, 1]$, while the plus extremals of g lie in $(z_1, z_2)\cup(z_3, z_4)$. Now we claim that if g has (respectively) a minus extremal in $[-1, z_1)$, a plus extremal in (z_1, z_2), a minus extremal in (z_2, z_3), a plus extremal in (z_3, z_4), or minus extremal in $(z_4, 1]$, then (respectively)

$$
\begin{aligned}
-2\delta \leqslant B(g)(-1) \\
B(g)(-1+\alpha) \leqslant 2\delta \\
-2\delta+\gamma \leqslant B(g)(-1+c) \\
B(g)(0) \qquad \leqslant 2\delta-\beta \\
-2\delta \leqslant B(g)(1).
\end{aligned} \tag{4.4}
$$

To prove the first inequality (the others are similar), note that if $x^* \in [-1, z_1)$ satisfies $g(x^*) - Bg(x^*) = -\|g - Bg\|$, then using (4.1) we have $f(-1) - Bg(-1) \leqslant f(x^*) - Bg(x^*)$ (since $|m| \leqslant$ the slope of the first segment of f) $= g(x^*) - Bg(x^*) - g(x^*) + f(x^*) \leqslant -(1-\delta) + \delta = -1 + 2\delta$, so $Bg(-1) \geqslant f(-1) + 1 - 2\delta = -2\delta$. Now there are five possible configurations for the points of an alternant for g. In the two cases which include minus extremals in both $[-1, z_1)$ and $[z_4, 1]$ we get from (4.2) and (4.4) that $\|Bg\| \leqslant 2\delta$, so $\|Bg - Bf\|/\|f - g\| \leqslant 2\delta/\delta = 2$. In the remaining three cases there will be a minus extremal in (z_2, z_3), and we compute $m = (Bg(-1+c) - Bg(-1))/c \geqslant (-2\delta + \gamma - 2\delta)/c = (\gamma - 4\delta)/c$ and $m = (Bg(1) - Bg(-1+c))/(2-c) \leqslant (2\delta - (-2\delta + \gamma))/(2-c) = (4\delta - \gamma)/(2-c)$, so $(\gamma - 4\delta)/c \leqslant -(\gamma - 4\delta)/(2-c)$, so $(\gamma - 4\delta)(1/c + 1/(2-c)) \leqslant 0$, so $\gamma \leqslant 4\delta$. It then follows from (4.3) that $\|Bg\| \leqslant 2(\beta/\gamma)\gamma + 6\delta \leqslant 8(\beta/\gamma)\delta + 6\delta$, so $\|Bg - Bf\|/\|f - g\| \leqslant 8(\beta/\gamma) + 6$, so this inequality holds in all cases, contradicting the assumption $\|Bg\|/\delta \to \infty$ and completing the proof that S has a uniform Lipschitz constant.

For an unbounded sequence of Lipschitz constants define piecewise linear functions f_α and g_α by

$$
g_\alpha(x) = \begin{cases} -1, & x = -1, 1 \\ 1 - \alpha^2, & x = -1 + \alpha \\ -1 - 2\alpha^2, & x = -1 + 2\alpha \\ 1 - \alpha & x = 0 \end{cases}
$$

and for f_α, let $c = 2\alpha$, $\gamma = \alpha^2$, and $\beta = \alpha$.

Then $Bg_\alpha(x) = -\alpha(x+1)$, $\|f_\alpha - g_\alpha\| = 3\alpha^2$ and

$$
\frac{\|Bg_\alpha\|}{\|f_\alpha - g_\alpha\|} = \frac{2}{3\alpha} \to \infty \qquad \text{as} \quad \alpha \to 0.
$$

Observe here that β/γ does not remain bounded as $\gamma \to 0$.

Remarks. (1) When $\beta = 1$ and $c = 2\alpha$ the previous functions f are almost the functions f in [7, Example 2].

(2) The example can also be modified for the situation $|E^0| \geqslant n + 1$.

The modified example just changes f on $[\frac{1}{2}, 1]$ making it identically -1 there. As before when β/γ is bounded there is a uniform Lipschitz constant; there is an example where β/γ is not bounded and there is no uniform Lipschitz constant.

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